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THE PROPAGATION OF ATMOSPHERIC ROSSBY GRAVITY WAVES IN LATITUDINALLY SHEARED ZONAL FLOWS

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CONTENTS

Using the β-plane approximation we formulate the equations which govern small perturbations in a rotating atmosphere and describe a wide class of possible wave motions, in the presence of a background zonal flow, ranging from 'moderately high' frequency acoustic-gravity-inertial waves to 'low' frequency planetary-scale (Rossby) waves. The discussion concentrates mainly on the propagation properties of Rossby waves in various types of latitudinally sheared zonal flows which occur at different heights and seasons in the earth's atmosphere. However, it is first shown that gravity waves in a latitudinally sheared zonal flow exhibit critical latitude behaviour where the 'intrinsic' wave frequency matches the Brunt–Väisälä frequency (in contrast to the case of gravity waves in a vertically sheared flow where a critical layer exists where the horizontal wave phase speed equals the flow speed) and that the wave behaviour near such a latitude is similar to that of Rossby waves in the vicinity of their critical latitudes which occur where the 'intrinsic' wave frequency approaches zero.

In the absence of zonal flow in the atmosphere the geometry of the planetary wave dispersion equation (which is described by a highly elongated ellipsoid in wavenumber vector space) implies that energy propagates almost parallel to the β-planes. This feature may provide a reason why there seems to be so little coupling between planetary scale motions in the lower and upper atmosphere. Planetary waves can be made to propagate eastward, as well as westward, if they are evanescent in the vertical direction.

The W.K.B. approximation, which provides an approximate description of wave propagation in slowly varying zonal wind shears, shows that the distortion of the wavenumber surface caused by the zonal flow controls the dependence of the wave amplitude on the zonal flow speed. In particular it follows that Rossby waves propagating into regions of strengthening westerlies are intensified in amplitude whereas those waves propagating into strengthening easterlies are diminished in amplitude. A classification of the various types of ray trajectories that arise in zonal flow profiles occurring in the Earth's atmosphere, such as jet-like variations of westerly or easterly zonal flow or a belt of westerlies bounded by a belt of easterlies, is given, and provides the conditions giving rise to such phenomena as critical latitude behaviour and wave trapping. In a westerly flow there is a tendency for the combined effects on wave propagation of jet-like variations of β and zonal flow speed to counteract each other, whereas in an easterly flow such variations tend to reinforce each other.

An examination of the reflexion and refraction of Rossby waves at a sharp jump in the zonal flow speed shows that under certain conditions wave amplification, or over-reflexion, can arise with the implication that the reflected wave can extract energy from the background streaming motion. On the other hand the wave behaviour near critical latitudes, which can be described in terms of a discontinuous jump in the 'wave invariant', shows that such latitudes can act as either wave *absorbers* (in which case the mean flow is accelerated there) or wave *emitters* (in which case the mean flow is decelerated there).

Introduction

In this paper we present a comprehensive discussion of the propagation properties of atmospheric waves, particularly planetary scale perturbations in a rotating atmosphere in the presence of background zonal winds, which includes such features as the dependence of the wave amplitude on the zonal flow speed, the classification of the various ray trajectories in different profiles of zonal flow, critical latitude phenomena which are accompanied by absorption or emission of waves energy, and wave amplification which may arise when waves are incident upon a sharp jump in the zonal flow speed.

Many properties of such waves are well established. For example the characteristic feature of westward phase propagation of planetary scale waves in an ocean has been studied, using the β-plane approximation, by Longuet-Higgins (1965) who also established the existence of a cut-off frequency, due to tidal effects, above which they cease to propagate. Lighthill (1967*a*) has examined the generation of Rossby waves on an ocean by travelling forcing effects. The dispersion equation describing the three-dimensional propagation of planetary waves has been derived by Lindzen (1967) by making use of β-planes. In the classical paper by Charney & Drazin (1961) it was shown that the effect on the vertical propagation of Rossby waves of zonal flows in the Earth's atmosphere was to make the dominant planetary waves evanescent in the vertical direction, so that the winds simulate a quasi-solid boundary which prevents the vertical propagation of the large flux of wave energy from the lower to the upper atmosphere. The wave behaviour at a critical latitude (i.e. a latitude where the zonal flow speed matches the wind speed in the case of Rossby waves) has been studied by Dickinson (1968) and Lindzen (1970) who have shown that a transfer of wave energy and momentum to the wind may take place at such latitudes in a fashion which is somewhat similar to the critical layers for gravity waves in a vertically sheared flow (see Booker & Bretherton 1967).

Using the β-plane approximation we formulate, in § 2, the equations governing the propagation of waves in a rotating atmosphere so as to provide a unified view of the various wave modes which include acoustic gravity-inertial waves as well as very low frequency planetary-scale (Rossby) waves. It is shown that gravity waves propagating in a latitudinally sheared zonal flow exhibit critical latitudes where the wave frequency, as measured in the rest frame of the fluid, matches the Brunt–Väisälä frequency, which contrasts with the case of Rossby waves and the more familiar case of gravity waves in a vertically sheared flow, in both of which a critical latitude or level is reached when the wave frequency in the rest frame approaches zero. This section is concluded by formulating the equation governing the latitudinal structure of planetary wave perturbations in preparation for the subsequent discussion, taken up in the next sections by making use of the W.K.B. and sharp boundary approximations, of their properties in the presence of zonal winds.

In § 3 a variety of results, for slowly varying media, are established by making extensive use of the geometrical properties of the wavenumber surface for planetary waves. First it is shown that in the absence of a zonal flow the geometry of the wavenumber surface (represented by a highly elongated ellipsoid) implies that vertical propagation of energy is inhibited by virtue of the nearly cylindrical character of the wave propagation which tends to constrain energy propagation to lie in β-planes. Thus this feature may provide a reason (other than the one given by Charney & Drazin (1961)) why there appears to be so little coupling between planetary scale motions in the lower and upper atmosphere. It is also shown that planetary waves can be made to propagate *eastward*, as well as westward, if they decay in the vertical direction. (Use is made of this property in § 4 in connection with the possibility of waves being amplified on reflexion from a vortex sheet.) The distortion of the wavenumber surface, which is caused by westerly and easterly zonal flows and is depicted in figures 2 and 3, controls the dependence of the wave amplitude on the zontal flow speed. It is established that Rossby waves propagating into regions of strengthening westerlies are intensified in amplitude whereas those waves propagating into regions of strengthening easterlies are diminished in amplitude. The former result is in qualitative agreement with the numerical analysis of Simmons (1974), who showed that the amplitude of a planetary wave disturbance tended to follow the magnitude of the westerly zonal flow. The remainder of this section is taken up with an investigation of the various ray trajectories that arise in (*a*) a westerly et stream, (*b*) an easterly jet stream, (*c*) an antisymmetric shear flow, (*d*) jet-like variations of β in a uniform wind. The results of the investigation are shown in figures 5–8, which illustrate the conditions giving rise to such diverse phenomena as critical latitude behaviour, wave trapping exclusion of waves from centre of a jet, and north to south (or vice versa) penetration of a jet.

The problem of reflexion and refraction of Rossby waves at a sharp jump in the zonal flow speed is studied in § 4. This situation is the opposite extreme to the W.K.B. approximation so that the results should be useful in discussing the reflexion process in a continuously varying zonal flow profile if the latitudinal wavelength greatly exceeds the length scale of the variation of the zonal shear flow. In particular we examine the conditions that can give rise to the amplitude of the reflected wave being greater than that of the incident wave. (We refer to this as wave amplification which implies that the reflected wave extracts energy from the background zonal flow.) This phenomenon occurs only if the waves are evanescent in the vertical direction so as to allow eastward propagating incident waves from a region of rest to be coupled with transmitted waves in a region of zonal flow that correspond to westward propagating waves that have been blown eastward by a sufficiently rapid westerly zonal flow. Such conditions, giving rise to wave amplification, can be interpreted in terms of the interaction of positive and negative energy waves in the same way as the over-reflexion of gravity waves from a vertical shear (see McKenzie 1972).

In the last section we investigate wave behaviour near critical latitudes, which correspond to regular singular points of the differential equation governing the latitudinal structure of the pressure perturbations. The form of this differential equation enables us to establish an invariant of the system – analogous to the wave action flux for gravity waves in a vertically sheared flow – which is independent of latitude except at critical latitudes where it discontinuously jumps from one constant to another. The proper matching procedure of the solutions across critical latitudes yields the jump in the invariant, which is accompanied by corresponding jumps in the northward fluxes of wave energy and zonal momentum. It is shown that a critical latitude is associated with either an *absorption* of wave energy and momentum (along with a corresponding acceleration of the mean flow there) or an *emission* of wave energy and momentum (which is accompanied by a corresponding deceleration of the mean flow at the critical latitude).

1. Governing equations and wave modes

Formulation of the equations

The equations of continuity, momentum and energy (the last in its dissipationless and therefore adiabatic form) for an inviscid fluid rotating with angular velocity $\boldsymbol{\Omega}$ under a gravitational acceleration $\boldsymbol{g}$ are

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\,\mathrm{div}\,\boldsymbol{u} = 0, \tag{1}$$

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\wedge\boldsymbol{u}\right) = -\nabla p+\rho\boldsymbol{g}, \tag{2}$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = c^2\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{3}$$

where $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{\Omega}$ the rate of rotation vector, ρ, p the density and pressure, c the sound speed, and $\mathrm{D}/\mathrm{D}t = \partial/\partial t+\boldsymbol{u}\cdot\nabla$ is the convective derivative.

If we assume a basic state in which the flow is purely zonal then this state satisfies the equation

$$\partial p_0/\partial z = -\rho_0(g-2\Omega_y U), \tag{4}$$

$$\partial p_0/\partial y = -\rho_0\, 2\Omega_z U, \tag{5}$$

in which we have taken local cartesian coordinates with x eastward, y northward, z vertical in the opposite direction to $\boldsymbol{g}$, and ρ_0, p_0 and U are respectively the density, pressure and zonal flow speed of the basic state.

When these equations are applied to the atmosphere of a rotating planet it is useful to employ the Rossby β-plane approximation (see, for example, Charney & Drazin 1961), which retains the important dynamical effect of the planet's curvature while neglecting its unimportant geometrical effect. By employing a β-plane approximation to a rotating isothermal atmosphere equations (4) and (5) yield

$$\rho_0 = \rho_{00} \exp\left(-\int \frac{dz}{H}\right) \exp\left(-\int \frac{dy}{L}\right), \tag{6}$$

in which ρ_{00} is a reference value of the density and H and L, the vertical and zonal scale lengths, are given by

$$H = RT/(g-hU), \quad L = RT/fU, \tag{7}$$

with

$$f = 2\Omega_z = f_0 + \beta y, \quad h = 2\Omega_y = h_0 - \alpha y,$$

$$(f_0, h_0) = 2\Omega(\sin\theta_0, \cos\theta_0), \quad (\beta, \alpha) = (2\Omega/a)(\cos\theta_0, \sin\theta_0);$$

θ_0 is the latitude on which β-plane is taken; a the radius of the planet; R the universal gas constant; and T the temperature. In studying perturbations of equations (1)–(3) about the basic state given by (4), (5) and (6) we introduce the 'normalized' perturbation velocity (see for example Eckhart 1960) (u^*, v^*, w^*), which is related to the perturbation velocity (u, v, w) by

$$(u^*, v^*, w^*) = \rho_0^{\frac{1}{2}}(u, v, w) \tag{8a}$$

and similarly for the perturbation pressure p and density ρ we have

$$(p^*, \rho^*) = \rho_0^{-\frac{1}{2}}(p, \rho). \tag{8b}$$

The dependence of the perturbation quantities on time t and west–east distance x is taken to be of the wave form $\exp\{i(\omega t - k_x x)\}$, where ω is the wave frequency and k_x, the zonal wavenumber, is given by

$$k_x = s/a,$$

where s is an integer to ensure periodicity. Linearization of equations (1), (2) and (3) about the basic state yields the following equations for the normalized perturbation quantities

$$i\hat{\omega}\rho^* - ik_x u^* + \left(\frac{\partial}{\partial y} - \frac{1}{2L}\right) v^* + \left(\frac{\partial}{\partial z} - \frac{1}{2H}\right) w^* = 0, \tag{9}$$

$$i\hat{\omega}u^* - \left(f - \frac{\partial U}{\partial y}\right) v^* + \left(h + \frac{\partial U}{\partial z}\right) w^* = ik_x p^*, \tag{10}$$

$$i\hat{\omega}v^* + fu^* = -\frac{\partial p^*}{\partial y} + \frac{p^*}{2L}, \tag{11}$$

$$i\hat{\omega}w^* - hu^* = -\frac{\partial p^*}{\partial z} + \frac{p^*}{2H} - g\rho^*, \tag{12}$$

$$i\hat{\omega}\left(\frac{p^*}{c^2} - \rho^*\right) - \left(\frac{RT}{c^2} - 1\right)\frac{v^*}{L} - \left(\frac{RT}{c^2} - 1\right)\frac{w^*}{H} = 0 \tag{13}$$

in which the Doppler shifted frequency $\hat{\omega}$ is given by the usual relation

$$\hat{\omega} = \omega - k_x U.$$

Three of the variables (ρ^*, w^* and u^*, say) can be eliminated in favour of the remaining two (v^* and p^*, say) so as to obtain a pair of coupled, partial differential equations which are analogous to the 'residual equations' studied by Eckhart (1960). Thus, use of (10), (12) and (13) to express u^*, ρ^* and w^* in terms of p^* and v^* yields

$$i\hat{\omega}w^* = \frac{\hat{\omega}^2}{\hat{\omega}^2-N^2-\omega_h^2}\left[-\frac{\partial}{\partial z}+\frac{1}{2H}-\frac{g}{c^2}+\frac{hk_x}{\hat{\omega}}\right]p^* + \frac{N^2}{\hat{\omega}^2-N^2-\omega_h^2}\left[\frac{H}{L}-\frac{h(f-\partial U/\partial y)}{N^2}\right]i\hat{\omega}v^*, \quad (14)$$

$$g\rho^* = \frac{N^2}{N^2-\hat{\omega}^2+\omega_h^2}\left\{\left[-\frac{\partial}{\partial z}+\frac{1}{2H}-\frac{(\hat{\omega}^2-\omega_h^2)}{N^2}\frac{g}{c^2}+\frac{hk_x}{\hat{\omega}}\right]p^* + \left[\frac{H}{L}\left(1-\frac{\omega_h^2}{\hat{\omega}^2}\right)-h\left(f-\frac{\partial U/\partial y}{\hat{\omega}}\right)\right]i\hat{\omega}v^*\right\}, \quad (15)$$

with u^* being given by (10) into which we substitute (14) and (15) for w^* and ρ^* and N, the Väisälä–Brunt frequency, and ω_h are given by

$$N^2 = \frac{g}{H}\left(1-\frac{RT}{c^2}\right); \quad \omega_h^2 = h\left(h+\frac{\partial U}{\partial z}\right).$$

The sound speed c is given by $c^2 = \gamma RT,$

where γ is the ratio of the specific heats. Therefore the limit of incompressible fluid motions is obtained by letting $\gamma \to \infty$ so that $c \to \infty$ while T remains finite.

Eliminating u^*, w^* and ρ^* from the continuity equation (9) yields the following relation between p^* and v^*,

$$\begin{aligned}
&\frac{i\hat{\omega}}{N^2-\hat{\omega}^2+\omega_h^2}\left[\frac{\partial^2 p^*}{\partial z^2}+\left(-\frac{1}{4H^2}+\frac{\hat{\omega}^2-\omega_h^2}{c^2}+(N^2-\hat{\omega}^2+\omega_h^2)\frac{k_x^2}{\hat{\omega}^2}\right)p^*+\frac{hk_x}{\hat{\omega}}\left(\frac{1}{H}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{hk_x}{\hat{\omega}}\right)p^*\right]\\
&\quad+\left[\frac{\partial p^*}{\partial z}+p^*\left(\frac{1}{H}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{hk_x}{\hat{\omega}}\right)\right]\frac{\partial}{\partial z}\left(\frac{i\hat{\omega}}{N^2-\hat{\omega}^2+\omega_h^2}\right)\\
&=\frac{\partial v^*}{\partial y}+\left[-\frac{k_x(f-\partial U/\partial y)}{\hat{\omega}}+\frac{\hat{\omega}^2-\omega_h^2}{N^2-\hat{\omega}^2+\omega_h^2}\frac{1}{L}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{h(f-\partial U/\partial y)}{N^2-\hat{\omega}^2+\omega_h^2}\frac{1}{H}\left(1-\frac{1}{\gamma}\right)\right]v^*\\
&\quad-\frac{N^2}{N^2-\hat{\omega}^2+\omega_h^2}\left[\left(\frac{H}{L}-\frac{h(f-\partial U/\partial y)}{N^2}\right)\left(\frac{\partial v^*}{\partial z}+\frac{k_x}{\hat{\omega}}(h+\partial U/\partial z)\,v^*\right)+\frac{h(f-\partial U/\partial y)}{2HN^2}v^*\right]\\
&\quad+v^*\frac{\partial}{\partial z}\left[\frac{N^2}{\hat{\omega}^2-N^2-\omega_h^2}\left(\frac{H}{L}-\frac{h(f-\partial U/\partial y)}{N^2}\right)\right].
\end{aligned} \quad (16)$$

The other relation between v^* and p^*, obtained by multiplying (11) by $i\hat{\omega}$ and eliminating u^* by means of (10) along with (14) and (15), is given by

$$\begin{aligned}
&\left\{f(f-\partial U/\partial y)-\hat{\omega}^2+\frac{f(h+\partial U/\partial z)\,N^2}{N^2-\hat{\omega}^2+\omega_h^2}\left(\frac{H}{L}-\frac{h(f-\partial U/\partial y)}{N^2}\right)\right\}v^*\\
&\quad=-i\hat{\omega}\left(\frac{\partial}{\partial y}+\frac{k_x f}{\hat{\omega}}-\frac{1}{2L}\right)p^*-\frac{if(h+\partial U/\partial z)\,\hat{\omega}}{N^2-\hat{\omega}^2+\omega_h^2}\left[\frac{\partial}{\partial z}-\frac{hk_x}{\hat{\omega}}-\frac{1}{H}\left(\frac{1}{2}-\frac{1}{\gamma}\right)\right]p^*.
\end{aligned} \quad (17)$$

This equation may be used to eliminate v^* from (16) to obtain a second-order partial differential equation for the normalized pressure perturbation p^*. The enormous mathematical difficulties associated with finding solutions to this equation, subject to appropriate boundary conditions, stem from its non-separability in its independent variables y and z and the presence of singular curves.

2. The various wave modes

From the physical point of view equations (16) and (17) describe a wide and rich class of possible wave motions in the presence of a background zonal flow, ranging from 'high' frequency acoustic waves at one extreme of the spectrum, through 'moderate to low-frequency' gravity-inertial waves, to very low frequency planetary-scale (Rossby) waves, at the other extreme. Therefore progress towards obtaining solutions to these equations is best made by adopting approximations appropriate to the dynamics of the particular wave mode. This is done in what follows so as to provide a somewhat unified view of the various wave modes.

(a) *Acoustic-gravity waves*

Let us first consider the equation governing the propagation of acoustic-gravity waves in a horizontal flow sheared vertically. This is obtained by putting $f = h = \partial U/\partial y = 0$ in equations (16) and (17) and using the latter to eliminate v^* in the former to yield

$$\frac{\partial^2 p^*}{\partial z^2}+\left(-\frac{1}{4H^2}+\frac{\hat{\omega}^2}{c^2}+k_x^2\frac{(N^2-\hat{\omega}^2)}{\hat{\omega}^2}\right)p^*+\frac{k_x}{\hat{\omega}}\frac{\partial U}{\partial z}\frac{\partial p^*}{\partial z}$$
$$+\left[\frac{\partial p^*}{\partial z}+\frac{p^*}{H}\left(\frac{1}{2}-\frac{1}{\gamma}\right)\right]\frac{(N^2-\hat{\omega}^2)}{\hat{\omega}}\frac{\partial}{\partial z}\left(\frac{\hat{\omega}}{N^2-\hat{\omega}^2}\right)-\frac{(N^2-\hat{\omega}^2)}{\hat{\omega}^2}\frac{\partial^2 p^*}{\partial y^2}=0. \tag{18}$$

In a uniform flow (i.e. $\partial U/\partial z = 0$) equation (18) admits plane wave solutions of the form $\exp\{-\mathrm{i}(k_y y+k_z z)\}$ provided the Doppler shifted frequency $\hat{\omega}$ and the wavenumber vector (k_x, k_y, k_z) satisfy the well known acoustic-gravity dispersion relation (see Eckhart 1960)

$$k_z^2 = \frac{\hat{\omega}^2-N_1^2}{c^2}+\frac{N^2-\hat{\omega}^2}{\hat{\omega}^2}(k_x^2+k_y^2) \tag{19}$$

in which N_1, the cut-off frequency for acoustic waves, is given by

$$N_1 = c/2H.$$

In a shear flow which varies sufficiently slowly in one spatial direction equation (19) may be regarded as the 'local' dispersion equation which determines either the local value of the vertical wavenumber k_z, in the case of a vertically sheared flow, or the local northward wavenumber k_y in the case of a flow sheared latitudinally. It is of some interest to construct the wavenumber surface (i.e. the surface in $\boldsymbol{k}$ space at a fixed frequency ω defined by equation (19)) for incompressible gravity waves (obtained from (19) by dropping the first term on the right-hand side) in a horizontal flow. Cross-sections of this surface taken through planes of k_z = constant and k_y = constant are sketched in figure 1 (*a*) and (*b*) respectively. These diagrams which are analogous to those constructed by Lighthill (1967 *a*) and which he used to discuss the various types of waves generated by a disturbance travelling with uniform speed, can also be used to derive the conditions for the existence of a critical level and for constructing ray (or wave-packet) trajectories in a sheared flow.

Within the framework of a W.K.B. analysis a critical level can be defined as a level to which a wave-packet, being neither reflected nor transmitted there, approaches asymptotically. The condition for the existence of a critical level for any type of wave motion has been examined by McKenzie (1972) who showed that if a cross-section of a wave normal surface possesses an asymptote a critical level can exist provided the properties of the medium vary in a direction parallel to the asymptote. If we apply this condition to gravity waves in a vertically sheared horizontal flow with a given ω, k_x and k_y then we see that all of the wavenumber surface cross-sections in figure 1 (*a*) are asymptotic to the line $k_x = \omega/U$ and hence a critical level can exist at a height z_c, where the flow speed equals the horizontal phase speed ω/k_x (in agreement with the original analysis of Bretherton (1966)). The dynamic aspects, as distinct from the foregoing kinematic ones, of this critical level behaviour can be examined by considering the matching

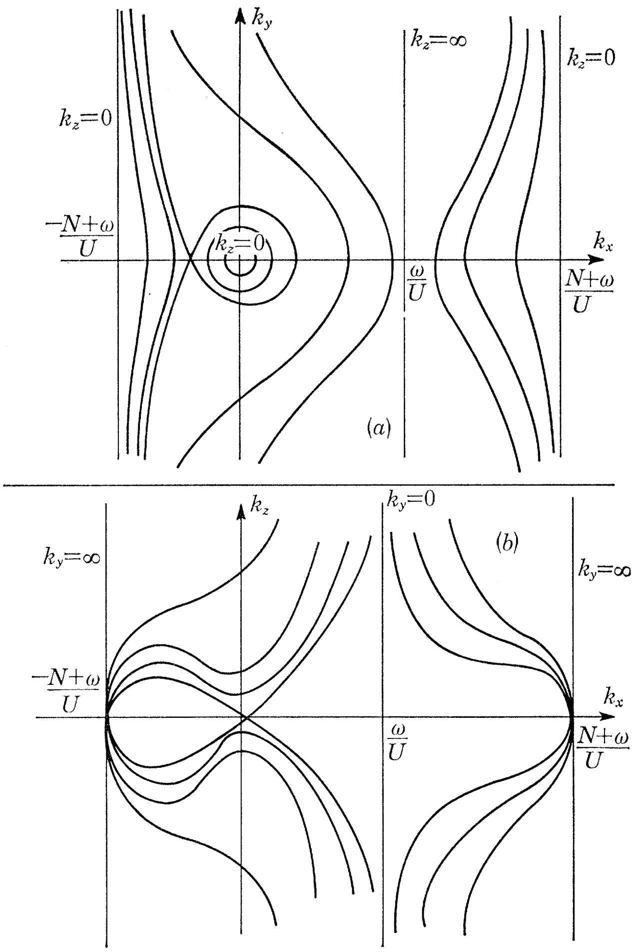

FIGURE 1. Cross-sections of the wavenumber surface for gravity waves in a uniform horizontal flow taken through planes of k_z = constant (figure 1 (*a*)) and k_y = constant (figure 1 (*b*)). In figure 1 (*a*) the wavenumbers curves are asymptotic to the line $k_x = \omega/U$, which corresponds to the wave frequency seen in the moving fluid Doppler-shifted to zero. The wavenumber curves in 1 (*b*) possess asymptotes at $k_x = (\omega \pm N)/U$ which correspond to the wave frequency measured in the rest frame of the fluid, being Doppler-shifted to the Brunt–Väisälä frequency.

procedure across the singularity occurring at $\hat{\omega} = 0$ in equation (18) and this has been carried out by several authors (see, for example, Booker & Bretherton 1967; Jones 1967) who have shown that the wave is strongly attenuated in crossing the critical level provided that the Richardson number of the shear is greater than about $\frac{1}{4}$.

Although the case of gravity waves in a latitudinally sheared flow does not appear to be appropriate to atmospheric situations it is instructive to examine it. Equation (19), which determines k_y for any given set of values of ω, k_x and k_z as a function of the flow speed $U(y)$, along with figure 1 (*b*) shows that cross-sections of the wavenumber surface taken through planes of k_z = constant, possess two asymptotes, namely

$$k_x = (\omega \pm N)/U, \tag{20}$$

which correspond to the frequency of the 'laboratory' frame $\hat{\omega}$ matching the Brunt–Väisälä frequency N. Thus applying McKenzie's condition we can deduce that a critical level can exist where the flow speed attains values such that equation (20) is satisfied. In contrast to the case of a vertically sheared flow these critical levels (or latitudes) may be approached from one side only in the fashion

$$x = |y - y_c|^{-\frac{1}{2}}.$$

This behaviour is related to the geometry of the wavenumber curves which are asymptotic to the lines $k_x = (\omega \pm N)/U$ from only one side, and to the nature of singularity occurring at $\hat{\omega} = \pm N$ in equation (18). Since we shall see later that this singularity is of the same type as planetary waves in a latitudinal shear we refrain from discussing it further here except to remark that the absorption of the wave at such a critical latitude will be *complete* since on one side the wave can propagate and the other it cannot.

(*b*) *Acoustic-gravity-inertial waves*

The effect of rotation on the wave modes described by equation (19) can be easily examined in the simplest case where there is no zonal flow (i.e. $U = 0$) and the only component of the Coriolis force is vertical and constant; equations (16) and (17) then yield

$$\frac{1}{N^2-\omega^2}\left[\frac{\partial^2 p^*}{\partial z^2} + \left(-\frac{1}{4H^2} + \frac{\omega^2}{c^2} + \frac{N^2-\omega^2}{\omega^2}k_x^2\right)p^*\right] + \frac{1}{f^2-\omega^2}\left[\frac{\partial^2 p^*}{\partial y^2} - \frac{k_x^2 f^2}{\omega^2}p^*\right] = 0. \tag{21}$$

This equation admits plane wave solutions provided that

$$k_z^2 = \frac{\omega^2 - N_1^2}{c^2} + \frac{N^2-\omega^2}{\omega^2 - f^2}(k_x^2 + k_y^2), \tag{22}$$

which is the dispersion equation for wave motions in an atmosphere rotating with constant angular frequency f about a vertical axis (see Eckhart 1960). If $N > f$, as is the case in the earth's atmosphere, vertical propagation is quenched for wave frequencies less than f. This result is modified if allowance is made for the variation of f with latitude using the β-plane approximation (see § 3 on planetary waves).

Equation (22) may be regarded as the local dispersion equation in a slowly varying shear flow provided that ω is replaced by $\hat{\omega}$. We then see that the effect of rotation in a vertically sheared flow is to 'split' the critical level, which occurred at $\hat{\omega} = 0$ in the absence of rotation, into two critical levels occurring at $\hat{\omega} = \pm f$ or $k_x = (\omega \mp f)/U$. The process of wave absorption at such levels has been discussed by Jones (1968) and more recently by Grimshaw (1975).

(*c*) *Wave modes, including the β-effect, in a zonal flow sheared latitudinally*

Finally we now turn to the formulation of the equation governing the propagation of wave modes including the 'β' effect. It is customary to neglect the horizontal component of the Coriolis term since it can only play a significant role extremely close to the equator where $f \to 0$.

We shall consider only the case of latitudinal shears so that with $h = \partial U/\partial z = 0$ we obtain, on substituting (16) into (17), the following equation for p^*

$$\frac{1}{N^2-\hat{\omega}^2}\left[\frac{\partial^2 p^*}{\partial z^2}+\left(-\frac{1}{4H^2}+\frac{\hat{\omega}^2}{c^2}+\frac{N^2-\hat{\omega}^2}{\hat{\omega}^2}k_x^2\right)p^*\right]+\frac{1}{\hat{\omega}}\left\{\frac{\partial}{\partial y}-\left[\frac{k_x(f-U')}{\hat{\omega}}+\frac{\hat{\omega}^2}{N^2-\hat{\omega}^2}\frac{1}{L}\left(\frac{1}{2}-\frac{1}{\gamma}\right)\right]\right\}$$
$$\times\left\{\frac{\hat{\omega}}{f(f-U')-\hat{\omega}^2}\left[\frac{\partial p^*}{\partial y}+\frac{k_x f}{\hat{\omega}}p^*-\frac{p^*}{2L}\right]\right\}$$
$$=\frac{N^2}{N^2-\hat{\omega}^2}\frac{H}{L}\frac{1}{f(f-U')-\hat{\omega}^2}\frac{\partial}{\partial z}\left(\frac{\partial p^*}{\partial y}+\frac{k_x f}{\hat{\omega}}p^*-\frac{p^*}{2L}\right) \tag{23}$$

in which, by comparison with equation (18) and (21), we can note the appearance of extra terms due to the variation of f and y (the β-effect) and U with y (the effect of latitudinal shear).

Since we have assumed $\partial U/\partial z = 0$ we can transform (23) to an ordinary differential equation by assuming vertical variations to be of the form $\exp\{-ik_z z\}$. To simplify this equation further we neglect terms of $O(L^{-1})$ compared with $k_x f/\hat{\omega}$ (the ratio of the former to the latter is

$$O\{(\hat{\omega}/f)/U/(10^4\,\mathrm{m\,s^{-1}})\},$$

which for even 'moderate' frequencies is very small in the Earth's atmosphere) so as to obtain the following equation governing the latitudinal structure:

$$\frac{d^2\phi}{dy^2}+(b-\tfrac{1}{4}a^2-\tfrac{1}{2}a')\,\phi=0 \tag{24}$$

in which the normalized pressure perturbation p^* has been renormalized such that

$$p^*=\phi\exp\left(-\tfrac{1}{2}\int a\,dy\right) \tag{25a}$$

with

$$a=\frac{-2\beta f+U'(2\hat{\omega}k_x-\beta)-fU''}{f(f-U')-\hat{\omega}^2}+ik_z\frac{H}{L}\frac{N^2}{N^2-\hat{\omega}^2} \tag{25b}$$

$$b=\frac{f(f-U')-\hat{\omega}^2}{N^2-\hat{\omega}^2}\left(-k_z^2-\frac{1}{4H^2}+\frac{\hat{\omega}^2}{c^2}\right)-k_x^2$$
$$-\frac{k_x}{\hat{\omega}}\left[\beta+\frac{2\hat{\omega}^2\beta+fU'(2\hat{\omega}k_x+\beta)-f^2U''}{f(f-U')-\hat{\omega}^2}+ik_z\frac{H}{L}\frac{fN^2}{N^2-\hat{\omega}^2}\right] \tag{25c}$$

in which we recall that

$$f=f_0+\beta y,\quad f_0=2\Omega\sin\theta_0,\quad \beta=(2\Omega/a)\cos\theta_0.$$

In slowly varying wind shears asymptotic (i.e. W.K.B.) solutions to (24) may be written in the form

$$\phi=A(y)\exp\{-iS(y)\}, \tag{26}$$

where $A(y)$, the slowly varying wave amplitude, and $S(y)$, the phase, are given by

$$\left(\frac{dS}{dy}\right)^2=X-\frac{1}{A}\frac{d^2A}{dy^2}, \tag{27a}$$

$$A=\frac{\text{constant}}{(dS/dy)^{\frac{1}{2}}}\exp\left(\int Y\,dy\right), \tag{27b}$$

in which X and Y are respectively the real and imaginary parts of the coefficient of ϕ in (24), i.e.

$$X=\mathrm{Re}\,(b-\tfrac{1}{4}a^2-\tfrac{1}{2}a'),\quad Y=\mathrm{Im}\,(b-\tfrac{1}{4}a^2-\tfrac{1}{2}a').$$

Since A is assumed to be slowly varying the term A''/A can be neglected on the right-hand side of (27a), which may be called the 'local' dispersion equation which determines the local latitudinal wavenumber k_y by the relation

$$k_y^2 = \left(\frac{\mathrm{d}S}{\mathrm{d}y}\right)^2 \approx X = \mathrm{Re}\,(b - \tfrac{1}{4}a^2 - \tfrac{1}{2}a'). \tag{28}$$

This equation contains the complete spectrum of wave modes in a latitudinal shear flow ranging from low-frequency planetary waves on the one hand to high-frequency acoustic waves on the other. In the next section we shall examine the properties of this equation under the planetary-wave approximations.

However, before proceeding to do this we note an interesting effect of the presence of the imaginary terms in (25b) and (25c); this is that in the case of reflexion from a discontinuity (or wall) located along some given latitude the reflected wavelength in the latitudinal direction would be different from the incident wavelength by an amount

$$2\pi\,\frac{N^2-\hat{\omega}^2}{N^2}\,\frac{L}{Hk_z}.$$

This follows from (25a) and assuming waves of the form $\exp\{\mp \mathrm{i}k_z z\}$ with the upper (lower) sign referring to the incident (reflected) wave.

3. The propagation of planetary-scale (Rossby) waves in a zonal flow

Using the W.K.B. approximation we investigate the propagation properties of planetary waves in: (a) a zonal flow with latitudinal shear including such flow profiles as easterly and westerly jet streams, and a belt of westerlies bounded by a belt of easterlies, all of which forms occur in the earth's atmosphere at different heights and seasons (Murgatroyd 1965; Lindzen & Siu-Shung Hong 1974); (b) a constant zonal wind but taking into account the variation of 'β' with latitude, with it attaining its maximum at the equator and falling off symmetrically on either side.

(a) *The planetary-waves dispersion equation*

Since we make extensive use of the geometrical properties of the planetary wavenumber surface in deducing the various types of ray trajectories that can arise in cases (a) and (b) above we first examine its properties. The planetary wave dispersion equation can be derived from equation (28) by making the following approximations:

(i) incompressible fluid motions ($\gamma \rightarrow \infty$);

(ii) $\hat{\omega} \ll N$–quasi-hydrostatic motions;

(iii) $\hat{\omega} \ll f$–quasi-geostrophic motions;

(iv) the imaginary part appearing on the right-hand side of (25b) is neglected so that $a \approx -2\beta/f$ since the ratio of it to the real part is of order $\{k_z H \tan\theta_0 \sin\theta_0\, U/(10^3\,\mathrm{m\,s^{-1}})\}$ which is small for wind speeds of the order of tens of metres/second:

(v) $(\beta^2/f^2)/(k_x\beta/\hat{\omega}) \sim (2\hat{\omega}/sf)\cot\theta_0 \ll 1$. From the dynamical viewpoint these approximations imply that when the wave motions are quasi-hydrostatic and geostrophic the rate of change of vorticity of a fluid element is balanced by the latitudinal change in planetary vorticity. With the above approximations (28) simplifies to

$$k_y^2+k_x^2+\frac{\beta k_x}{\hat{\omega}}=-\frac{f^2}{N^2}\left(k_z^2+\frac{1}{4H^2}\right) \tag{29a}$$

or alternatively
$$k_y^2+\left(k_x+\frac{\beta}{2\hat{\omega}}\right)^2=\frac{\beta^2}{4\hat{\omega}^2}-\frac{f^2}{N^2}\left(k_z^2+\frac{1}{4H^2}\right). \tag{29b}$$

Comparing (29a) with the gravity-inertial wave dispersion equation (22) we see that $\hat{\omega}$ has been dropped from the terms $\hat{\omega}^2-f^2$ and $N^2-\hat{\omega}^2$; N_1^2/c^2 has been replaced by its equivalent form $\frac{1}{4}H^{-2}$ and the β effect appears through the term $\beta k_x/\hat{\omega}$. This form of the dispersion equation shows that the propagation of atmospheric Rossby waves in horizontal directions is analogous to two-dimensional propagation in an ocean of depth d, where d^{-1} is equivalent to $H(k_z^2+\frac{1}{4}H^{-2})$ (see, for example, Longuet-Higgins 1965 or Lighthill 1967a).

In the absence of a zonal flow and at a fixed frequency ω, equation (29) represents in $\boldsymbol{k}$ space an ellipsoid of revolution, with axis the line parallel to the k_z axis displaced $-\beta/2\omega$ units along the k_x axis, and ratio of major to minor axis N/f. Cross-sections taken through planes k_z = constant are concentric circles of radii

$$\left\{\frac{\beta^2}{4\omega^2}-\frac{f^2}{N^2}\left(k_z^2+\frac{1}{4H^2}\right)\right\}^{\frac{1}{2}} \quad \text{with centre} \quad (k_x, k_y)=\left(\frac{-\beta}{2\omega}, 0\right).$$

The well-known result that phase propagation of planetary waves is *westward* follows from the fact that such cross-sections lie entirely in the half space $k_x < 0$ (since we have assumed the waves to be of the form $\exp\{i(\omega t-k_x x)\}$).

The value of the vertical wavenumber at which the radius of the circular cross-section vanishes corresponds to the ultimate k_z permitting northward (or southward) phase propagation and provides the spatial counterpart of the cut-off frequency, given by $(\beta^2 gd/4f^2)^{\frac{1}{2}}$, above which Rossby waves cease to propagate in an ocean of depth d (Longuet-Higgins 1965). In the Earth's atmosphere the minimum cut-off frequency, obtained by putting $k_z=0$, normalized to the angular frequency of rotation is $(NH/\Omega a)\cot\theta_0$ which corresponds to a period of $2\times 2\tan\theta_0$ days (in which we have taken $2\pi/N = 5$ minutes and $H = 10$ km). An interesting propagation feature arising from the existence of a large value of N/f, which is more than 10^2 in the Earth's atmosphere, is that since the ellipsoid is extremely elongated along its axis of revolution and can therefore be approximated by a cylinder, wave packets, or rays, are constrained to propagate very close to horizontal planes. This follows from the fact that the direction of a ray is parallel to the normal to the wavenumber surface. Thus vertical propagation of energy is inhibited by virtue of the nearly cylindrical character of the wave propagation, which tends to confine energy propagation to lie in β-planes. This feature may provide another reason why there appears to be so little coupling between planetary scale motions in the lower and upper atmosphere. The classical reason (Charney & Drazin 1961) is that for typical zonal wind profiles the dominant planetary waves (taken to be characterized by a longitudinal wavenumber of 2, corresponding to two oceans and two continents, and meridional wavenumber somewhat greater than 2 corresponding to a wavelength somewhat less than 2×10^4 km) Rossby waves are evanescent in the vertical direction (i.e. $k_z^2 < 0$).

It is therefore interesting to consider the propagation effects of negative values of k_z^2 (i.e. wave decaying in the vertical direction). From equation (29) we see that planetary waves can be made to propagate eastward, as well as westward, for sufficiently large values of $-k_z^2$, such that

$$-k_z^2 > \beta^2/4\omega^2 - f^2/4H^2N^2.$$

In addition such evanescence in the vertical direction permits northward phase propagation at frequencies above the cut-off frequency $(NH/\Omega a)\cos\theta_0$ (day)$^{-1}$.

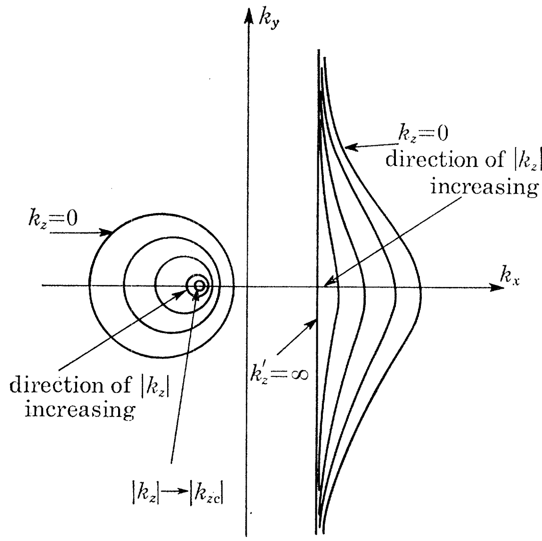

FIGURE 2. The planetary wave's number surface in a westerly ($U > 0$) zonal wind depicted as curves in the (k_y, k_x) plane representing different cross-sections taken through planes of k_z = constant. At a given k_z the curves cut the k_x axis at three points given by the roots of the zeros of the numerator of the right-hand side of equation (30*a*). The k_{zc} indicated in the figure is the ultimate vertical wavenumber permitting northward phase propagation. Thus equation (30*a*) represents two surfaces, one a closed surface (a 'distorted ellipsoid') corresponding to westward ($k_x < 0$) phase propagation, and the other an open surface, a plane with an eastward facing bump, corresponding to westward propagating waves that have been blown eastward ($k_x > 0$) by the westerly zonal flow.

In order to consider the distortion of the wavenumber surface introduced by a zonal wind flowing with speed U we write the dispersion relation (29) in the form

$$k_y^2 = \frac{k_x(k_x-k_+)(k_x-k_-) - (f^2/N^2)(k_z^2+1/4H^2)\{(\omega/U)-k_x)\}}{(\omega/U)-k_x} \tag{30a}$$

in which

$$k_\pm = (\omega/2U)\{1 \pm (1+4\beta U/\omega^2)^{\frac{1}{2}}\}. \tag{30b}$$

We use the above form to construct the wavenumber surface as cross-sections taken through planes of k_z = constant. Figure 2 illustrates cross-section in a westerly (i.e. $U > 0$) zonal flow. The wavenumber surface is made up of two parts, one part consists of a closed surface (a 'distorted ellipsoid') corresponding to westward (i.e. $k_x < 0$) phase propagation, and the other part, an open surface which may be visualized as the plane $k_x = \omega/U$ with a bump facing eastward, corresponds to those westward propagating waves that have been 'blown' *eastward* by the westerly flow. In an easterly (i.e. $U < 0$) flow the geometry of the wavenumber surface depends upon whether or not the magnitude of the flow speed is less or greater than $\omega^2/4\beta$. If $|U| < \omega^2/4\beta$, it consists of a closed surface (the 'distorted ellipsoid') and an open surface which may be thought of as the plane $k_x = \omega/U$ with a bump facing eastward (see figure 3*a*). Figure (3*b*) illustrates the case $|U| > \omega^2/4\beta$ when the closed surface becomes embedded inside the plane with the bump.

These surfaces are the three-dimensional counterpart of the wavenumber curves for two-dimensional Rossby waves generated by an oscillating forced effect travelling with uniform speed U (Lighthill 1967*a*).

In the limit $(f/N) \to 0$ the surfaces become cylindrical. For example, in figure 2, westward propagation is represented by the distorted circular cylinder one of whose generators is the line $k_x = 0$, and eastward propagation is represented by the plane $k_x = \omega/U$ with an infinitely long eastward facing bump running parallel to the k_z axis. For simplicity we use this approximation to the wavenumber surface in the subsequent discussion of the various ray trajectories in different profiles of zonal flow.

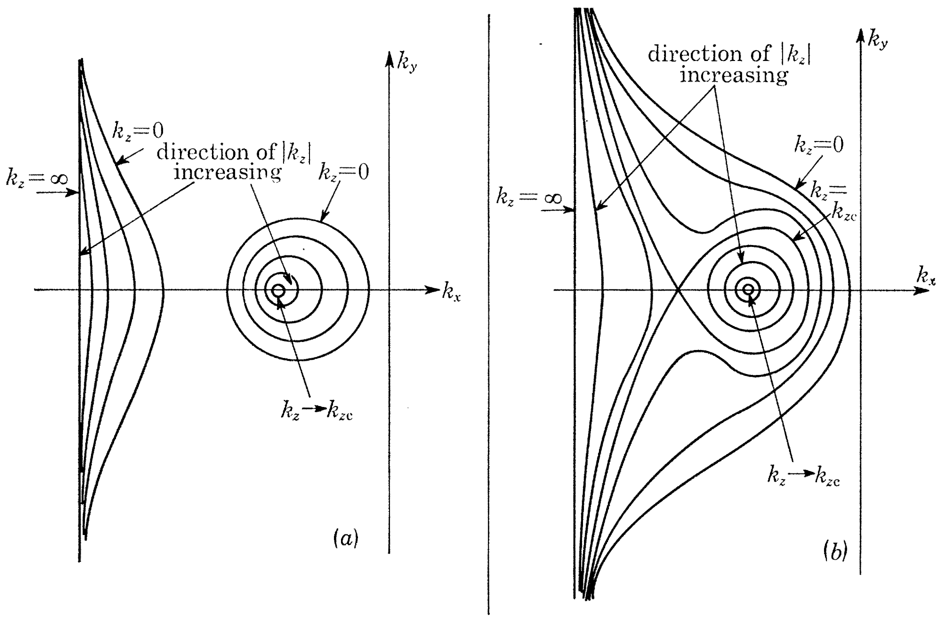

FIGURE 3. Cross-section of the wavenumber surface in an easterly ($U < 0$) zonal flow. (*a*) Wavenumber curves for $|U| < \omega^2/4\beta$. (*b*) If $|U| > \omega^2/4\beta$ the closed surface becomes embedded in the plane with a bump.

(*b*) *The dependence of the wave amplitude on zonal flow-speed*

Under the approximations leading to the planetary wave dispersion relation (29) the variation of the wave amplitude A, within the W.K.B. approximation, is given by (27*b*) which may be written

$$A\,|k_y|^{\frac{1}{2}} = \text{constant}. \tag{31}$$

This equation states that the wave amplitude at any flow-speed (or latitude) is inversely proportional to the square root of the local latitudinal wavenumber at that speed (or latitude). Since a wave packet, in a zonal flow sheared latitudinally, propagates in such a manner that ω, k_x and k_z are conserved along its path we can follow the change in wave amplitude by determining the change in k_y from one speed to another.

In figure 4 we have sketched the wavenumber curves at a fixed frequency ω for various values of westerly ($U > 0$) and easterly ($U < 0$) zonal wind speeds. This figure shows that for a given fixed value of k_x (three of which are indicated by vertical lines intersecting the various wavenumber curves) the magnitude of the local latitudinal wavenumber, k_y, descreases for U positive and increasing, but increases for U negative and increasing in magnitude. Therefore from (31) we deduce that Rossby waves propagating into regions of strengthening westerlies are intensified in amplitude while those propagating into regions of strengthening easterlies are diminished in amplitude (this result has been generalized by Eltayeb & McKenzie (1976) to the case of two-dimensional hydromagnetic Rossby waves).

Near reflexion points (i.e. $k_y \to 0$) the W.K.B. solution given by (31) breaks down and the reflexion process is then described by a Stokes equation which yields an Airy function behaviour for the wave amplitude variation (see Budden 1961). The behaviour near critical latitudes (where $k_y \to \infty$) is described by obtaining asymptotic solutions near the singular point of the full wave equation and these are developed in § 5.

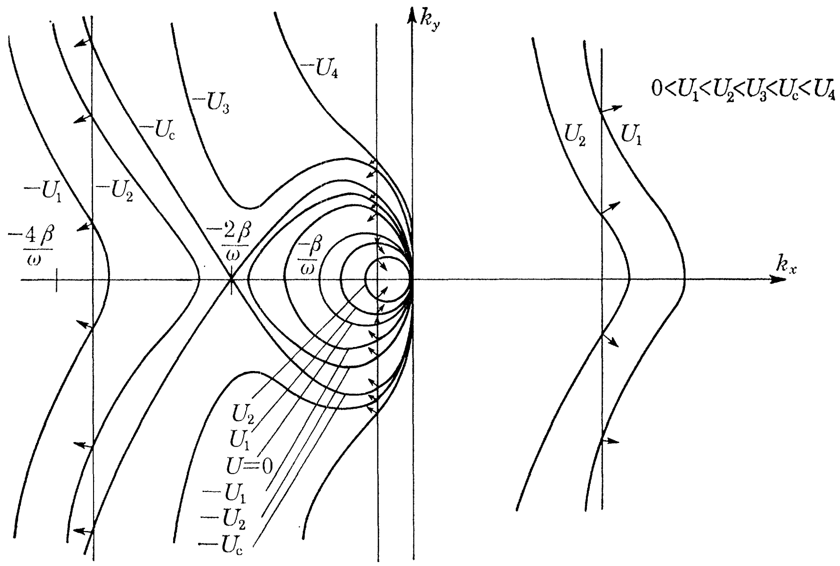

FIGURE 4. The locus of real wavenumbers in the (k_y, k_y) plane at a fixed frequency for increasing values of westerly (U_1 and U_2) and easterly ($-U_1$, $-U_2$, $-U_c$, $-U_4$) zonal flow speeds. The small arrows drawn normal to the locus of wavenumber at points of intersection with lines of k_x = constant indicate the direction of a wave packet (ray) at each flow speed. This diagram illustrates that strengthening westerlies enhance wave amplitudes whereas strengthening easterlies diminish them (see equation (31) and the subsequent text.)

We note that if the flow were sheared vertically (rather than latitudinally) a similar result would hold, namely strengthening westerlies enhance wave amplitudes whereas strengthening easterlies diminish them. This statement follows from the result of a W.K.B. analysis which yields $A\,|k_z|^{\frac{1}{2}}$ = constant, and an examination similar to that shown in figure 4 of the changes of shape of cross-sections of the wavenumber surface taken through planes of k_y = constant.

(c) *Ray trajectories in zonal flows*

Here we discuss the various types of ray trajectories that can arise in different profiles of sheared zonal flow. This is done by sketching the locus of wavenumbers in the (k_y, k_x) plane at successive latitudes and following the direction of small arrows (representing the ray direction) drawn normal to the locus where it is intersected by a line k_x = constant at each latitude. (This construction has been used extensively to examine ray paths, for example by Lighthill (1967 *a*) and McKenzie (1973).) The locii of wavenumbers possesses an asymptote at $k_x = \omega/U$ (i.e. where the longitudinal phase speed matches the zonal flow speed) where k_y reaches to infinity in the manner

$$k_y \sim |\omega/U - k_x|^{-\frac{1}{2}}.$$

Therefore in accordance with the aforementioned condition (McKenzie 1973) a critical 'latitude' y_c, can exist where $U(y_c) = \omega/k_x$ and the ray approaches the latitude in the fashion

$$x = |y - y_c|^{-\frac{1}{2}}.$$

Since the main results of this section are contained in figures 5–8 particular attention should be paid to these diagrams.

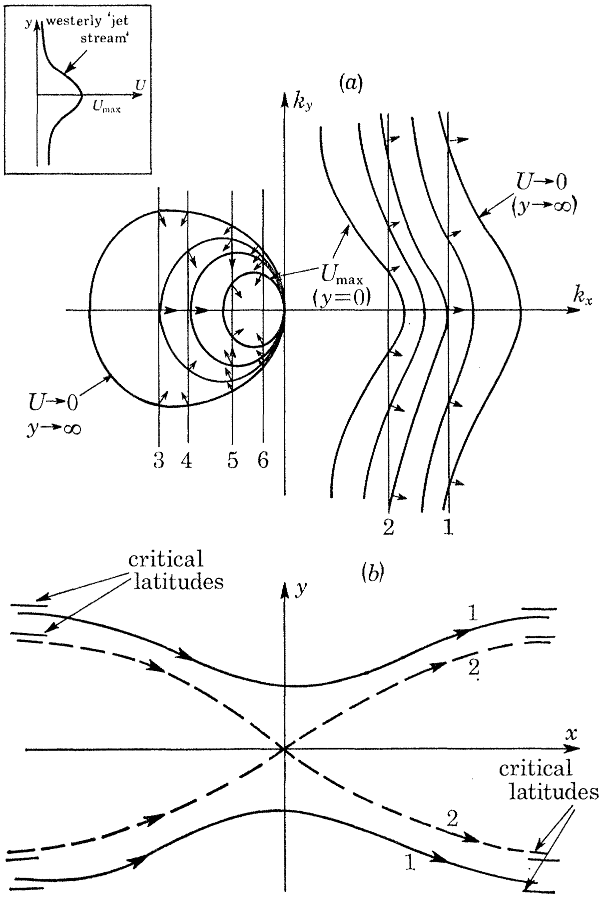

FIGURE 5. (*a*) The locus of wavenumbers in the (k_y, k_x) plane at successive latitudes in a westerly jet-like variation of zonal flow. For westward $(k_x < 0)$ phase propagation the smallest closed loop corresponds to the centre of the jet while the largest loop corresponds to the wings of the jet. For eastward $(k_x > 0)$ phase propagation the open branches move steadily to the right as the zonal flow speed decreases (i.e. as $|y|$ increases). The various types of ray trajectories can be constructed by following the direction of the arrows, drawn normal to the locus of wavenumbers where it intersects a given line of k_x = constant, at each flow speed or latitude.

The six ray trajectories sketched in (*b*), (*c*) and (*d*) correspond to the lines of k_x = constant labelled 1–6.

(*b*) The ray paths 1 and 2 exhibit critical latitudes where the zonal flow speed matches the longitudinal phase speed. Ray 1 is reflected northward (or southward) from latitudes where $k_x = k_+$ (or equivalently where $U = c(1+\beta c/\omega^2)$, $c = \omega/k_x$). The conditions for each type of ray to arise are:

Ray 1 corresponds to $k_x > k_+(0)$ or $U(0) > c(1+\beta c/\omega^2)$.

Ray 2 corresponds to $\omega/U(0) < k_x < k_+(0)$ or $c(1+\beta c/\omega^2) > U(0) > c$.

(i) *Westerly jet stream*

We first consider wave propagation in a westerly jet-like variation of the zonal flow. For convenience the centre of the jet is taken at $y = 0$, where $U = U_{max}$ and the flow speed falls off symmetrically on either side of the centre. Figure 5 (*a*) shows the geometrical construction which reveals the various ray paths sketched in figures 5 (*b*), (*c*) and (*d*).

Roughly speaking there are three classes of ray trajectories:

Class I. Rays exhibiting critical latitude behaviour (see rays labelled 1 and 2 in figure 5 (*b*) corresponding to the lines k_x = constant labelled 1 and 2 in figure 5 (*a*));

Class II. Rays that are reflected before reaching the centre of the jet (see rays 3 and 4 in figure 5 (*c*) corresponding to the lines labelled 3 and 4 of k_x = constant; ray 2 also falls into this class);

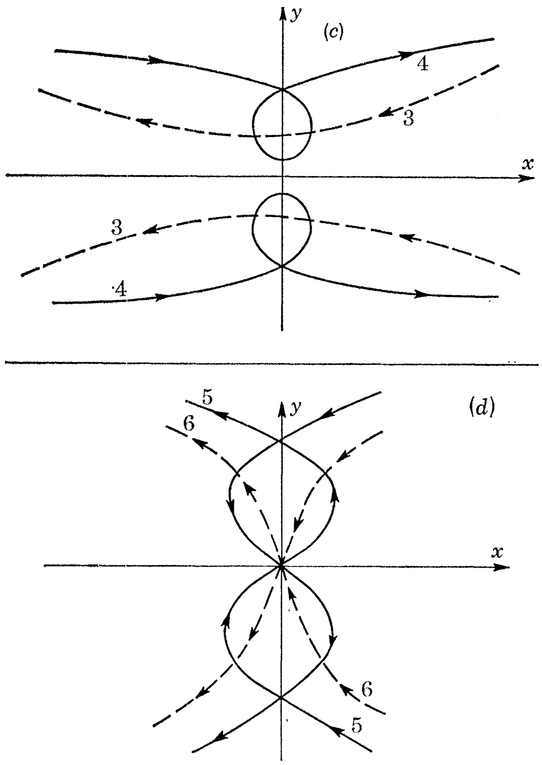

FIGURE 5. (*c*) The ray paths corresponding to the lines of k_x = constant labelled 3 and 4 in (*a*).
Ray 3 corresponds to values of k_x such that $-\beta/\omega < k_x < -\beta/2\omega$.
Ray 4 corresponds to $-\beta/2\omega < k_x < k_-(0)$.
Note that ray 4 corresponds to westward energy transport whereas ray 3 gives eastward energy propagation.
(*d*) Ray paths corresponding to lines, labelled 5 and 6 in (*a*). The values of k_x are such that $k_x < k_-(0)$. These rays can penetrate through the centre of the jet.

Class III. Rays that penetrate through the centre of the jet (rays 5 and 6, corresponding to the lines of k_x = constant labelled 5 and 6 in figure 5 (*a*), shown in figure 5 (*d*) fall into this category). The conditions giving rise to each type of ray trajectory are given in the legend of figure 5.

The source of each ray may be any point on the ray path since integration of the ray equation, namely

$$dx/dy = -\partial k_y/\partial k_x,$$

introduces an arbitrary constant which is available for fixing the location of the source. For convenience in drawing the ray paths we have chosen the origin in each case in such a way that the ray paths are symmetric about the y-axis.

A close inspection of figure 5 (*a*) will reveal how the different ray paths are intimately linked with the changes in shape of the wavenumber curves, induced by changes in flow speed, and its intersections with the different lines of k_x = constant.

(ii) *Easterly jet stream*

Next, in a similar fashion to above, we investigate ray trajectories which are possible in an easterly jet-like variation of the zonal flow. Figure 6(*a*) has been constructed for the situation where the flow speed at the centre of the jet exceeds the 'critical' speed $\omega^2/4\beta$. The ray paths corresponding to the intersections of the lines, labelled 1 to 5, of k_x = constant with the wave-number curves are sketched in figures 6(*b*), (*c*) and (*d*). In this case rays of the type defined above as Class I and III arise (see figures 6(*d*) and (*b*) respectively) but Class II type rays are missing.

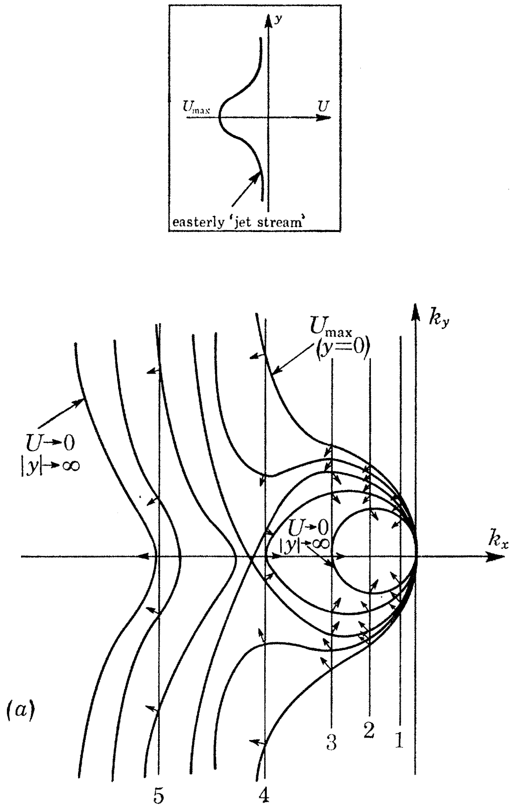

FIGURE 6(*a*). For description see opposite.

Instead we obtain a fourth class, namely:

Class IV. Rays trapped around the centre of the jet (see ray 3 in figure 6(*c*) and ray 6 in figure 6(*f*));

and a fifth class defined as follows:

Class V. Rays that 'emerge' from a critical latitude and transport energy towards the wings of the jet (see rays labelled 1′ and 2′ in figure 6(*e*)). In fact the rays 1′ and 2′ are special cases of the rays labelled 1 and 2, shown in figure 6(*b*), that arise if the flow speed at the centre of the jet is sufficiently large to make the lines k_x = constant (labelled 1 and 2 in figure 6(*a*)) correspond to an asymptote of the wavenumber curve at some latitude. We note that if the source of such rays were near the critical latitude the ray propagates *away* from this latitude and transports energy towards the wings of the jet.

From the intuitive point of view it would be natural to think of this type of critical latitude as a possible 'emitter', rather than absorber, of waves. For example if the source of a disturbance were located somewhere near the centre of the jet, from which rays 1′ and 2′ are excluded, the matching of this disturbance across the critical latitude into the region north (or south) of the critical latitude should be represented by a wave carrying energy *away* from the critical latitude. This idea is examined in sections where it is shown that such critical latitudes can act as wave *emitters*.

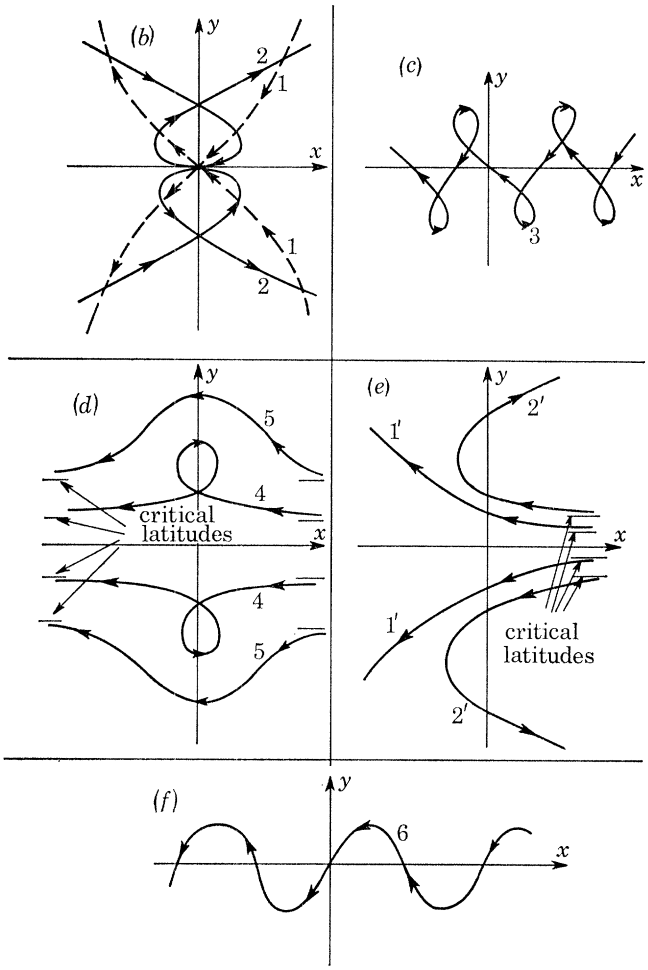

FIGURE 6. (*a*) Wavenumber curves in an easterly jet (drawn for $|U(0)| > \omega^2/4\beta$). The ray trajectories corresponding to the lines of $k_x =$ constant marked 1 to 5 are shown in (*b*), (*c*) and (*d*).

(*b*) Values of $k_x > -\beta/2\omega$ yields ray 1. The range $-\beta/\omega < k_x < -\beta/2\omega$ gives rays of the type labelled 2 which is the only ray in an easterly jet that can transport energy from west to east.

(*c*) The trapped ray 3, corresponding to $\omega/U(0) < k_x < -\beta/\omega$ is reflected from latitudes where $k_x = k_-$, or where $U = c(1+\beta c/\omega^2)$.

(*d*) Rays exhibiting critical latitudes.

Ray 4 corresponds to the range $-2\beta/\omega < k_x < \omega/U(0)$.

Ray 5 corresponds to values of $k_x < -2\beta/\omega$.

(*e*) If the zonal speed is sufficiently great (i.e. $|U| > 2\omega^2/\beta$) that the lines marked 1 and 2 in figure 6 (*a*) of $k_x =$ constant are asymptotes of the wavenumber curves at some latitude, we obtain the rays marked 1′ and 2′. These rays can 'emerge' from a critical latitude.

(*f*) The trapped ray, labelled 6, cannot be constructed from figure 6 (*a*) since it occurs only if $|U(0)| < \omega^2/4\beta$ and k_x lies in the range $\omega/U(0) < k_x < k_+(0)$.

(iii) *Anti-symmetric shear flow*

We now consider wave propagation in an anti-symmetric shear flow represented by a belt of increasing westerlies in the region $y > 0$ bounded below (i.e. in $y < 0$) by a belt of increasing easterlies. The ray paths can be constructed in the same fashion as above. In this case figure 5 (*a*) would apply to the region $y > 0$ with the direction of y increasing in that figure reversed since the flow speed increases with latitude, while figure 6 (*a*) would apply to the belt of easterlies, with the direction of y decreasing reversed. Figures 7 (*b*) and (*c*) illustrate respectively eastward and westward propagating rays, exhibiting critical latitudes, confined either to the belt of westerlies (figure 7 *b*) or the belt of easterlies (figure 7 *c*). Figures 7 (*d*) and (*e*) show that rays with sources in the easterlies are reflected, either from the belt of westerlies (figure 7 *d*) or before reaching the

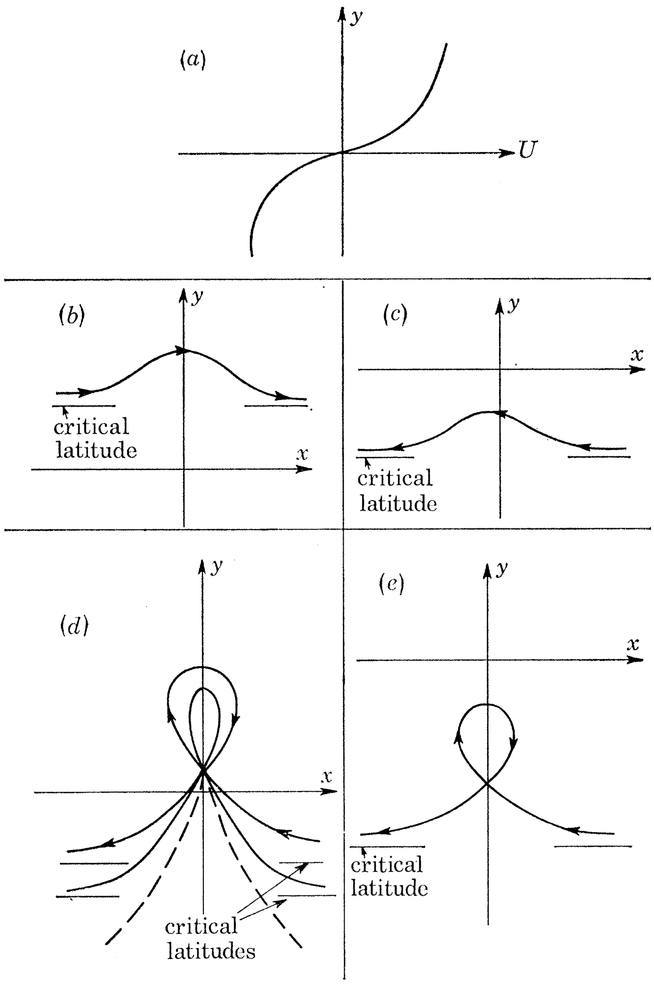

FIGURE 7. Ray paths in a belt of increasing westerlies (in $y > 0$) bounded below by a belt of increasing easterlies (in $y < 0$) (figure (*a*)).

(*b*) The eastward propagating ray in the region of westerlies; exhibiting critical latitude behaviour, corresponds to values of $k_x > \omega/U_{max}$ (where U_{max} is the maximum speed attained by the westerlies).

(*c*) The westward propagation ray, captured at a critical latitude in the easterlies, arises if $k_x < -\omega/|U_{max}|$.

(*d*) Westward propagating ray reflected from westerlies and captured at a critical latitude in the easterlies. The broken curve joining on to the full represents the ray path if the easterly wind does not reach sufficient strength for a critical latitude to develop.

(*e*) Westward propagating ray reflected from regions of weak easterlies and captured in strong easterlies.

zero wind line (figure 7*e*), back into increasing easterly winds where they can be eventually 'captured' at a critical latitude if the wind speed is sufficiently great.

(iv) *Effect of jet-like variations of β in a uniform zonal flow*

Finally we examine the effect of latitudinal variations of β on the propagation of Rossby waves in a uniform wind. First let us consider a jet-like variation of β with latitude in a uniform westerly wind. The relevant geometrical construction for the ray paths is shown in figure 8 (*a*) and the ray paths are sketched in figures 8 (*b*), (*c*) and (*d*). These show that the main effect of jet-like variations of β is to trap the waves about the equator (where β attains its maximum value β_m. Figure 8 (*a*) can also be used to deduce that waves are intensified as they propagate northward (southward) into regions of increasing β. (This follows from W.K.B. waves amplitude variation given by equation (31).)

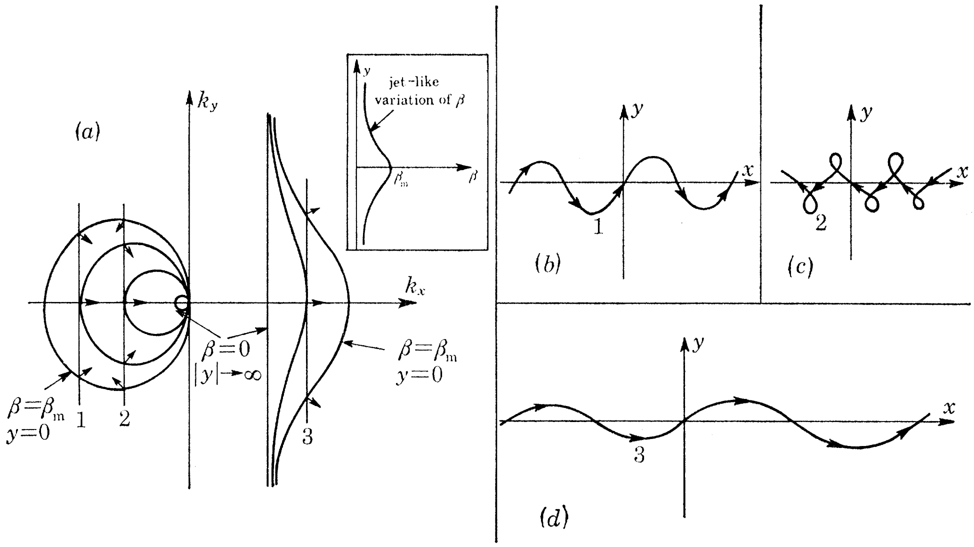

FIGURE 8. (*a*) Wavenumber curves, in a uniform westerly, at different latitudes for a jet-like variation of β. The arrows drawn normal to these curves at the points of intersection with a line k_x = constant indicate the ray direction.
(*b*), (*c*) and (*d*) are respectively the ray paths corresponding to the lines labelled 1, 2 and 3 in (*a*). All rays are trapped in a jet-like variation of β.

The combined effect of variation in the zonal flow-speed and β in the propagation of Rossby waves will depend on the precise profiles of U and β as functions of latitude. However, qualitatively we can see, by comparing the rays in figures 5 (*c*) and (*d*) with their counterparts shown in figures 8 (*b*) and (*c*), that jet-like variation in flow speed and β have opposite effects. This tendency for the combined effects of jet-like variations of U and β to counteract each other arises because for a fixed β the closed loop of the wavenumber curve (representing westward phase propagation) expands as U decreases (or $|y|$ increases), whereas for a fixed U the closed loop shrinks as β decreases (or $|y|$ increases). Similar conclusions can be drawn for eastward propagating waves (represented by the open wavenumber curve) by inspecting the ray labelled 3 in figure 8 (*d*) and the rays marked 1 and 2, depicted in figure 5 (*b*). Similarly we can determine the ray paths that arise in a uniform easterly, flow due to jet-like variations of $\beta(y)$. If $\beta(0) < -\omega^2/4U$, rays of the type labelled 1 and 2 shown in 8 (*b*) and (*c*) are possible. The ray corresponding to the open branch of the wavenumber curve (which now corresponds to westward phase propagation) is similar to that shown in 8 (*d*) except that the direction of the arrow should be reversed to indicate westward energy propagation. Therefore in an easterly flow the combined effects of jet-like variations in U and β on Rossby waves tend to reinforce each other, because such variations induce similar changes in the shape of the wavenumber curves.

4. Reflexion and refraction of Rossby waves at a vortex sheet

In this section we examine the opposite extreme to the W.K.B. approximation, namely the reflexion and refraction of Rossby waves at a discontinuity in the zonal flow speed. Such a study yields results to which the case of waves with latitudinal wavelength well in excess of the length scale of variation of the zonal flow should be asymptotic.

Consider two uniform zonal flows separated by a vortex sheet located at the plane $y = 0$. The undisturbed flow velocities $U_1\hat{\boldsymbol{x}}$ and $U_2\hat{\boldsymbol{x}}$ are tangential to the sheet, where subscripts 1 and 2 refer respectively to the regions $y > 0$ and $y < 0$. A wave incident, from $y < 0$, upon the sheet gives rise to a reflected wave, a transmitted wave, and a wave-like distortion of the sheet. The boundary conditions, namely continuity of displacement and pressure balance at the distorted sheet, determine the amplitudes of the reflected and transmitted waves.

In regions 1 and 2 the northward perturbation velocity may be written in the form

$$\left.\begin{aligned} V_1^* &= \exp(\mathrm{i}k_{y1}y) + R\exp(-\mathrm{i}k_{y1}y) \quad (y<0),\\ V_2^* &= T\exp(\mathrm{i}k_{y2}y) \quad (y>0),\end{aligned}\right\} \tag{32}$$

where R and T are respectively the wave amplitude reflexion and transmission coefficients corresponding to an incident wave of unit amplitude. We have assumed perturbations of the form $\exp\{\mathrm{i}(\omega t - k_x x - k_z z)\}$ so that the latitudinal wavenumbers k_{yi} $(i = 1, 2)$ satisfy the planetary wave dispersion equation

$$k_{yi}^2 = -k_x^2 - \frac{\beta k_x}{\hat{\omega}_i} - \frac{f^2}{N^2}(k_z^2 + \tfrac{1}{4}H^{-2}), \tag{33}$$

$$\hat{\omega}_i = \omega - k_x U_i.$$

The pressure perturbations may be written in terms of the northward velocity perturbations by using equation (17), which, on using the planetary wave approximations, gives

$$f^2V^* = -\mathrm{i}\hat{\omega}\left(\frac{\partial p^*}{\partial y} + \frac{k_x f}{\hat{\omega}}p^*\right). \tag{34}$$

The laws of reflexion and refraction follow from continuity of ω, k_x and k_z across the vortex sheet. (These are illustrated in figure 8 which is drawn for conditions, to be discussed below, giving rise to wave amplification.) The normal component of the wave vector satisfies equation (33) on either side of the sheet. Its sign must be chosen in such a way that in region 1 the energy flux of the incident wave is directed towards the sheet, whereas in region 2 the energy flux of the transmitted wave must diverge from the sheet. If $k_{y2}^2 < 0$ the ambiguity in the sign is removed by insisting that the amplitude of the transmitted wave decays into region 2. In this case total reflexion (i.e. $R = 1$) occurs as equation (38) and (39) demonstrate since then k_{y2} is purely imaginary. On linearization the kinematic boundary condition, namely continuity of displacement becomes

$$V_1^*/\hat{\omega}_1 = V_2^*/\hat{\omega}_2 \tag{35}$$

since the density is continuous at $y = 0$ although its derivative is not, since the latitudinal length scales L_1 and L_2 are different (see equation (7)). On substituting (32) into (35) we obtain

$$\frac{1+R}{\hat{\omega}_1} = \frac{T}{\hat{\omega}_2}. \tag{36}$$

The dynamic boundary condition ensuring pressure balance across the sheet yields

$$\frac{\mathrm{i}(1+R)\,U_1}{f\hat{\omega}_1} + \frac{1}{(\hat{\omega}_1 k_{y1} - \mathrm{i}k_x f)} + \frac{R}{(-\hat{\omega}_1 k_{y1} - \mathrm{i}k_x f)} = \frac{T}{(\hat{\omega}_2 k_{y2} - \mathrm{i}k_x f)} + \frac{\mathrm{i}TU_2}{f\hat{\omega}_2}. \tag{37}$$

The reflexion coefficient, following from (36) and (37), is given by

$$R = \frac{A - \mathrm{i}B}{C + \mathrm{i}B} \tag{38}$$

in which

$$\left.\begin{aligned} A &= \frac{\hat{\omega}_2^2 k_{y2}}{\hat{\omega}_2^2 k_{y2}^2 + k_x^2 f^2} - \frac{\hat{\omega}_1^2 k_{y1}}{\hat{\omega}_1^2 k_{y1} + k_x^2 f^2}, \\ B &= k_x f\left\{\frac{\hat{\omega}_1}{\hat{\omega}_1^2 k_{y1}^2 + k_x^2 f^2} - \frac{\hat{\omega}_2}{\hat{\omega}_2^2 k_{y2}^2 + k_x^2 f^2}\right\} - \frac{U_1 - U_2}{f}, \\ C &= \frac{-\hat{\omega}_2^2 k_{y2}}{\hat{\omega}_2^2 k_{y2}^2 + k_x^2 f^2} - \frac{\hat{\omega}_1^2 k_{y1}}{\hat{\omega}_1^2 k_{y1}^2 + k_x^2 f^2}. \end{aligned}\right\} \tag{39}$$

Equation (38) shows that wave amplication (i.e. $|R| > 1$), sometimes referred to as over-reflexion (Jones 1968), can occur if $A^2 > C^2$, which simplifies to

$$k_{y2} k_{y1} < 0. \tag{40}$$

This condition states that wave amplication can arise only if it is possible to find circumstances in which the northward (and therefore normal to the sheet) components of the incident and transmitted wavenumbers are of opposite sign.

Without loss of generality we assume region 1 ($y < 0$) to be at rest, with respect to which region 2 is flowing with speed U. Waves in region 1 are characterized by the property that their northward components of group and phase velocities are of opposite sign; thus an incident wave carrying energy towards the sheet (i.e. northwards) corresponds to southward phase propagation. Hence wave amplification can arise only if the transmitted wave is characterized by northward phase, as well as ray propagation. An examination of the dispersion equation assuming k_z real, shows that if the flow is easterly all waves are characterized by their northward components of phase and ray velocities being of opposite sign. Therefore wave amplification is not possible. When the flow is westerly waves belonging to the open surface (i.e. those that have been blown eastward by the westerly flow) are characterized by their northward components of phase and ray velocities being of the same sign. However, it is not possible for an incident wave in region 1 to be transmitted as an eastward wave in region 2 since k_x (which must be continuous across the sheet) in region 1 is negative whereas in region 2 it is positive for waves belonging to the open surface. Thus again wave amplification is not possible.

Qualitatively we can see that the conditions for wave amplification can be fulfilled only if waves in region 1 were capable of propagating eastward (i.e. $k_x > 0$) and the corresponding transmitted wave in region 2 were one belonging to the open wavenumber surface caused by the westerly flow. In § 3 (I) we have seen that if Rossby waves are evanescent in the vertical direction (i.e. $k_z^2 < 0$) eastward (i.e. $k_x > 0$) phase propagation is possible in the absence of zonal flow. Putting $k_z^2 = -K_z^2$ in (29*b*) gives

$$k_y^2 + \left(k_x + \frac{\beta}{2\hat{\omega}}\right)^2 = \frac{\beta^2}{4\hat{\omega}^2} + \frac{f^2}{N^2}\left(K_z^2 - \frac{1}{4H^2}\right), \tag{41}$$

which shows that eastward propagation is possible if $K_z > \frac{1}{2}H^{-1}$. Figure 9 depicts geometrically the necessary and sufficient conditions that give rise to wave amplification. These may be written in the form

$$\frac{\omega}{U} < \mathrm{OA} < k_\mathrm{m} \tag{42a}$$

in which

$$\mathrm{OA} = \left\{\frac{\beta^2}{4\hat{\omega}^2} + \frac{f^2}{N^2}\left(K_z^2 - \frac{1}{4H^2}\right)\right\}^{\frac{1}{2}} - \frac{\beta}{2\hat{\omega}} \tag{42b}$$

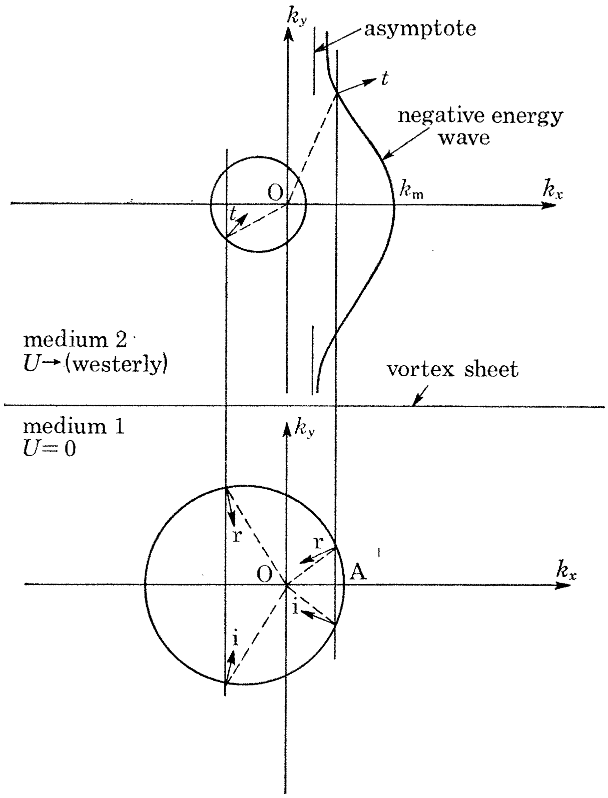

FIGURE 9. Geometrical construction illustrating Snell's laws of reflexion and refraction of Rossby waves at a vortex sheet. Incident waves propagating eastward such that $\omega/U < k_x < k_\mathrm{m}$ are amplified on reflexion from the sheet (see equation (42) and the accompanying text).

and k_m is the larger of the two positive roots of the following cubic equation for k_x,

$$k_x(k_x - k_+)(k_x - k_-) = (f^2/N^2)(K_z^2 - 1/4H^2)(k_x - \omega/U) \tag{42c}$$

in which $k_\pm$ are given by (30*b*). The first inequality in relation (42*a*), namely $\omega/U < \mathrm{OA}$, ensures that condition (40) is satisfied, while the second inequality, $\mathrm{OA} < k_\mathrm{m}$ ensures that k_{y2} is real since values of $k_x < k_\mathrm{m}$ would yield imaginary values of k_{y2} and would correspond to total reflexion. The first inequality requires that the westerly flow speed exceeds the minimum eastward phase speed in region 1, whereas the second inequality sets an upper limit on the flow speed to ensure a propagating (rather than evanescent) transmitted wave. The relations (40) and (42) may be interpreted as meaning that wave amplification can occur only if the energy density of the incident wave is of opposite sign to the energy density of the transmitted wave (see McKenzie 1973; Eltayeb & McKenzie 1976). If a wave carries a deficiency of energy as measured in the laboratory frame it can be referred to as a negative energy wave. This normally arises if the speed of the streaming motion relative to the laboratory frame exceeds the wave's phase speed in the direction of streaming. Thus in the situation depicted in figure 9 the incident and reflected wave are *positive* energy waves whereas the transmitted wave, belonging to the open branch of the wavenumber curve, is a *negative* energy wave. Therefore if conditions (42) are satisfied an incident wave with a horizontal phase speed, $c = \omega/k_x$, lying in the range $U > c > \omega/k_m$, will be amplified on reflexion from the vortex sheets thereby extracting energy from the streaming motion.

5. Wave behaviour near critical latitudes for Rossby and gravity waves in a latitudinally sheared flow (physical implications)

In §§ 3 and 4 we studied the propagation of Rossby waves using, respectively, the slowly varying (W.K.B.) and sharp boundary approximations, neither of which are capable of adequately describing the proper behaviour near critical latitudes. We now consider the wave behaviour and its physical implications near such latitudes which correspond to regular singular points of the differential equation governing the latitudinal structure. This analysis is similar to the one developed by Booker & Bretherton (1967) for gravity waves in a vertically sheared flow and is an extension of Dickinson's (1968) work on the propagation of planetary waves through waves through weak westerlies.

(a) *The wave invariant*

The latitudinal structure of the 're-normalized' pressure perturbation ϕ is determined by equation (24) which, on making the 'planetary-wave' approximations, may be written

$$\mathrm{d}^2\phi/\mathrm{d}y^2 + g(y)\,\phi = 0, \tag{43a}$$

in which
$$g(y) = b - \tfrac{1}{4}a^2 - \tfrac{1}{2}a' \tag{43b}$$

in which we neglect the small imaginary parts.

If the coefficient of ϕ in 43(*a*), i.e. $g(y)$, is real (i.e. k_z^2 real) it follows from the mathematical properties of second-order differential equations of the above type that the Wronskian is a constant; thus if the function ϕ is solution of (43*a*) and $\bar{\phi}$ its complex conjugate, then this fact can be used to establish an invariant of the system (i.e. quantity independent of latitude y), $\mathscr{A}$, say, given by

$$\mathscr{A} = \mathrm{Im}\left(\bar{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}y}\right). \tag{44}$$

The wave invariant $\mathscr{A}$, which is analogous to the wave action flux for gravity waves in a vertical shear flow, is constant except at critical latitudes where it is discontinous.

(b) *Matching solutions across singular points*

Regular singularities of equation (43*a*) occur at $\hat{\omega} = 0$, for 'low-frequency' Rossby waves and at $\hat{\omega} = \pm N$ for 'high-frequency' gravity waves. (Although the equation for ϕ is also singular at $\hat{\omega}^2 = f(f-U')$ this is not discussed since the corresponding equation for p^* is not singular there when we take account of 25 (a)). In the neighbourhood of a critical latitude $y = y_c$ equation (43) may be approximated by

$$\frac{d^2\phi}{dy^2} + \frac{\alpha}{y - y_c}\phi = 0 \tag{45a}$$

in which, for Rossby waves,

$$\alpha \equiv \frac{\beta - U''(y_c)}{U'(y_c)\,(1 - U'(y_c)/f)}, \quad U(y_c) = \omega/k_x \tag{45b}$$

whereas, for gravity waves,

$$\alpha \equiv \frac{-(U(y_c)/U'(y_c))\,(k_z^2 + 1/4H^2)}{2(1 \mp \omega/N)}, \quad U(y_c) = \frac{\omega \pm N}{k_x}. \tag{45c}$$

with the upper (lower) sign referring to $\omega = N(-N)$. The method of Frobenius yields the following series solutions for ϕ,

$$\phi = Au_1 + Bu_2, \tag{46a}$$

where A and B are arbitrary constants and u_1 and u_2, the two independent solutions of (45a), are given by

$$u_1 = (y - y_c) \sum_{n=0}^{\infty} \frac{(-\alpha(y - y_c))^n}{(n!)^2\,(n+1)},$$

$$u_2 = u_1 \int dy/u_1^2. \tag{46b}$$

The wave nature is brought out by the alternative solution which may be written

$$\phi = (y - y_c)^{\frac{1}{2}} \{CH_1^{(1)}[2\{\alpha(y - y_c)\}^{\frac{1}{2}}] + DH_1^{(2)}[2\{\alpha(y - y_c)\}^{\frac{1}{2}}]\} \tag{47}$$

in which C and D are constants, and $H_1^{(1)}$ and $H_1^{(2)}$ are respectively the first-order Hankel functions of the first and second kinds. The asymptotic form of the Hankel functions illustrate the propagating or evanescent nature, obtained by the pressure perturbation at points remote from the critical latitude, which can be joined to W.K.B. type solutions.

In matching the solutions across the singular points we follow the procedure outlined by Miles (1961) and adopted by Booker & Bretherton (1967) to the problem of critical levels for gravity waves in a vertically sheared flow. This procedure simulates causality by allowing the frequency ω to have a small imaginary part, i.e. $\omega = \omega_r + i\omega_i$ so that a disturbance with time dependence $\exp i\omega t$ is slowly growing in time by assuming that $\omega_i < 0$. The quantity $y - y_c$ in the differential equation is then replaced by z, say, where

$$z = y - y_c - i\omega_i/k_x U'(y_c) \tag{48a}$$

$$= |z|\,e^{i\Phi}, \quad \Phi = \arctan\left[\frac{|\omega_i|/k_x U'(y_c)}{(y - y_c)}\right]. \tag{48b}$$

For small values of z the series solution (46a) approximates to

$$\phi = A(1 - \alpha z \log z) + Bz. \tag{49}$$

Since ω_i is small $\arg z (= \Phi)$ goes from a small positive (negative) value, for values of $y > y_c$ to $\pi(-\pi)$ for values of $y < y_c$, when $k_x U' > 0(< 0)$. Thus the connection of the solutions, obtained by taking the limit $\omega_i \to 0$, on either side of the critical latitude is given by

$$\phi = A[1 - \alpha(y - y_c)\log(y - y_c)] + B(y - y_c) \quad (y > y_c)$$

$$= A[1 - \alpha(y - y_c)\,(\log|y - y_c| \pm i\pi)] + B(y - y_c) \quad (y < y_c) \tag{50}$$

in which the upper or lower sign appearing in front of $i\pi$ is taken according as $k_x U' > 0$ or < 0.

These approximate solutions for ϕ near $y = y_c$ can be used to establish the following values for the invariant $\mathscr{A}$, given by (44), above and below the critical latitude:

$$\mathscr{A}_{\text{above}} = \operatorname{Im}(\bar{A}B) \quad (y > y_c),$$
$$\mathscr{A}_{\text{below}} = \operatorname{Im}(\bar{A}B) \mp \alpha\pi\,|A|^2 \quad (y < y_c), \tag{51}$$

where the upper (lower) sign is chosen if $k_x U' > 0$ (< 0). It follows that the jump in the invariant across a critical latitude is given by

$$\mathscr{A}_{\text{above}} - \mathscr{A}_{\text{below}} = \pm\alpha\pi\,|A|^2. \tag{52}$$

(c) *Physical implications*

The northward fluxes of energy, F, and zonal momentum, M, per unit mass are given by

$$F = \overline{p^*v^*} + \rho U u^* v^*,$$
$$M = \overline{u^*v^*}, \tag{53}$$

where the overbar denotes a time average. The zonal and northward components of the velocity are given in terms of the pressure by

$$\left.\begin{aligned} u^* &= (\hat{\omega}^2 - f(f-U'))^{-1}\left[(f-U')\frac{\partial p^*}{\partial y} + \hat{\omega}k_x p^*\right], \\ v^* &= \mathrm{i}(\hat{\omega}^2 - f(f-U'))^{-1}\left[\hat{\omega}\frac{\partial p^*}{\partial y} + k_x f p^*\right]. \end{aligned}\right\} \tag{54}$$

Substituting (54) into (53) and using (25*a*), which is written in form

$$p^* = h\phi, \quad h \equiv \exp\left(-\int \tfrac{1}{2}a\,\mathrm{d}y\right) = \{f(f-U') - \hat{\omega}^2\}^{\frac{1}{2}} \tag{55}$$

we find, by using (44), that F and M can be written in terms of the wave invariant $\mathscr{A}$ as follows,

$$\left.\begin{aligned} F &= \hat{\omega}\mathscr{A}, \\ M &= k_x\mathscr{A}. \end{aligned}\right\} \tag{56}$$

These equations show that the 'classical' relation $F = (\hat{\omega}/k_x)\,M$ holds and confirm that $\mathscr{A}$ should be identified as the wave action flux.

We see therefore that a jump in the wave action flux across a critical latitude is accompanied by corresponding jumps in the northward fluxes of energy and momentum. Thus a critical latitude is associated with either an *absorption* of wave energy and momentum (with a corresponding acceleration of the mean flow there) or an *emission* of wave energy and momentum (accompanied by a corresponding deceleration of the mean flow). In other words, a critical latitude can behave as either a wave absorber or a wave emitter. Consider first a situation illustrating the somewhat more uncommon view of a critical latitude which behaves like a wave emitter that corresponds to ray propagation of the types labelled 1′ and 2′, in figure 6 (*e*), in an easterly jet. We take $y > 0$ (so that $U' > 0$), choose $k_x < 0$ and sufficiently small that the ray path represents an 'escape' trajectory from the critical latitude. From the physical point of view such a path implies that if the source of the disturbance is north of the critical latitude the main wave propagates northward away from the critical latitude. If, however, the source of a disturbance is south of this critical latitude the disturbance must be matched across this latitude in accordance with equations

(51)–(53). For large y the radiation condition requires that the disturbance takes the form of an outgoing wave so that we may write

$$\phi = C \exp(\mathrm{i}k_y y)$$

where C, the wave amplitude at infinity, can be calculated only by solving a particular problem since it is related to the amplitude of the initial disturbance, the matching procedure across the singularity, and the structure of the zonal shear. Thus the wave invariant in the region $y > y_c$ can be written

$$\mathscr{A} = k_y |C|^2, \quad y > y_c,$$

which is positive since k_y, which is determined from the dispersion relation, must be positive to ensure that the wave carries energy northward. If the zonal flow is symmetric (an easterly jet) so that a critical latitude exists in the region $y < 0$ at some $y = -y_c$, the disturbance must also be matched across $y = -y_c$ with the corresponding wave propagating southward towards $y = -\infty$, and the wave invariant in this region may then be written

$$\mathscr{A} = -k_y |D|^2, \quad y < -y_c \quad (k_y > 0),$$

where D is the wave amplitude at $y = -\infty$. The relative strengths of C and D depend on the particular problem. However the main point is that the critical latitudes behave as wave 'emitters' (since the disturbance within the latitudes is of the evanescent type) and that this emission is accompanied by a deceleration of the mean flow near such latitudes, so as to account for the radiation of energy and momentum to $|y| = \infty$.

The more usual case in which critical latitudes behave as wave absorbers is illustrated by, for example, rays 4 and 5 in figure 6(d). In this case if the source of the disturbance is north of the critical latitude the main wave propagates northward until it is reflected southward and is eventually absorbed at the critical latitude. In the case when the source is south of the critical latitude (in the evanescent region) a wave is first emitted from the critical latitude, propagates northward until it is reflected and eventually gets absorbed back into the critical latitude. Thus in such a case the critical latitude behaves like a 'source-sink' combination for wave generation.

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